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A new ring-shaped potential and its dynamical invariance algebra

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Abstract. A new ring-shaped potential, obtained by replacing the Coulomb part of the Hartmann potential by a harmonic oscillator term, is investigated. The Schrödinger equation is solved in spherical, circular cylindrical, prolate and oblate spheroidal coordinates. As in the case of the Hartmann potential, the 'accidental' degeneracies occurring in the spectrum are shown to be due to an su(2) dynamical invariance algebra. This establishes a close connection between both ring-shaped potentials.

1. Introduction

In the spectra of some quantum mechanical systems, there appear 'accidental' degeneracies, i.e. degeneracies not connected with obvious geometrical symmetries of the Hamiltonian, such as rotational invariance in the case of central potentials. Such 'accidental' degeneracies are often due to the existence of a dynamical invariance algebra. Two well known examples are the hydrogen atom and the oscillator, whose degeneracies were explained a long time ago by an so(4) and su(3) dynamical invariance algebra respectively (Fock 1935, Bargmann 1936, Jauch and Hill 1940). Ever since, dynamical invariance algebras have been determined for many systems exhibiting 'accidental' degeneracies (see, for example, Louck *et al* 1973, Moshinsky *et al* 1975, Moshinsky and Patera 1975, Quesne 1986). For some other systems, however, the explanation of 'accidental' degeneracies by the existence of a dynamical invariance algebra has been questioned (Moshinsky and Quesne 1983 and references therein).

Quite recently, much work has been devoted to the Hartmann potential (1972) due to its applications to ring-shaped molecules. This potential results from adding a repulsive potential proportional to $(r \sin\theta)^{-2}$ to an attractive Coulomb one. The Schrödinger equation is separable in both spherical and parabolic rotational coordinates (Hartmann 1972, Gerry 1986), and can also be solved via the Kustaanheimo-Stiefel transformation (Kibler and Négadi 1984b). The resulting discrete spectrum exhibits 'accidental' degeneracies due to an su(2) dynamical invariance algebra (Kibler and Winternitz 1987).

The purpose of the present paper is to investigate a new ring-shaped potential, obtained by replacing the Coulomb part of the Hartmann potential by a harmonic oscillator term. This potential is defined in § 2. In § 3, the corresponding energy spectrum, wavefunctions and integrals of motion are obtained. In § 4, the 'accidental'

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degeneracies appearing in the spectrum are explained by the existence of an su(2) dynamical invariance algebra. Finally, § 5 contains the conclusion.

2. The potential

In spherical coordinates r, θ, φ, the Hartmann potential is defined by

$$V_q = \eta\sigma^2\left(\frac{2a_0}{r} - q\eta\frac{a_0^2}{r^2\sin^2\theta}\right)\varepsilon_0 \tag{2.1}$$

where a_0, ε_0 stand for the Bohr radius and the ground-state energy of the hydrogen atom and η, σ, q are three dimensionless positive parameters. The potential originally considered by Hartmann (1972) corresponds to $q=1$. For fixed r, V_1 is minimum in the equatorial plane $(\theta=\frac{1}{2}\pi)$. In this plane, its minimum is given by $V_{10}=\sigma^2\varepsilon_0$ and is obtained for $r_0=\eta a_0$. The introduction of the extra parameter q makes it possible to obtain the Coulomb potential for a hydrogen-like atom as a limiting case of the potential (2.1) by taking $q=0$ and $\eta\sigma^2=Z$ (Kibler and Winternitz 1987).

Let us now consider the potential

$$V_q = \frac{\sigma^2}{2\eta^2}\left(\frac{r^2}{b_0^2} + q\eta^4\frac{b_0^2}{r^2\sin^2\theta}\right)\varepsilon_0 \tag{2.2}$$

where b_0 and ε_0 are defined in terms of a mass parameter μ and some fixed angular frequency ω_0 by $b_0=(\hbar/\mu\omega_0)^{1/2}$, $\varepsilon_0=\frac{1}{2}\hbar\omega_0$, and η, σ, q are three dimensionless positive parameters. For $q=0$ and $\sigma^2=2\eta^2$, we obtain the harmonic oscillator potential $V_0=\frac{1}{2}\mu\omega_0^2r^2$. For $q=1$ and fixed r, the potential (2.2) is minimum in the equatorial plane $(\theta=\frac{1}{2}\pi)$. In this plane, its minimum is given by $V_{10}=\sigma^2\varepsilon_0$, and is obtained for $r_0=\eta b_0$. Hence, V_1 is a ring-shaped potential, rather similar to the Hartmann one, except that it is a confining potential and has therefore only a discrete spectrum. In the next section, we shall determine the latter, as well as the corresponding wavefunctions.

3. Energy spectrum and wavefunctions

We shall henceforth adopt units such that $\hbar=\mu=\omega_0=1$ and use the shorthand notation $\gamma=\sigma/(\eta\sqrt{2})$. The Schrödinger equation for the potential (2.2) is

$$H\psi = E\psi \tag{3.1}$$

where

$$H = \frac{1}{2}\left[-\Delta + \gamma^2\left(r^2 + \frac{q\eta^4}{r^2\sin^2\theta}\right)\right] \tag{3.2}$$

Δ is the Laplacian, and E denotes the energy in units of $\hbar\omega_0$.

According to Makarov *et al* (1967), for any potential of the class

$$V = \alpha r^2 + \frac{\beta}{z^2} + \frac{h(y/x)}{x^2+y^2} \tag{3.3}$$

where α and β are any two constants and $h(y/x)$ is any function of y/x, the Schrödinger equation separates not only in spherical coordinates, but also in circular cylindrical, prolate spheroidal and oblate spheroidal ones. The potential in (3.2) belongs to this class and corresponds to

$$\alpha = \tfrac{1}{2}\gamma^2 \qquad \beta = 0 \qquad h = \tfrac{1}{2}q\eta^4\gamma^2. \tag{3.4}$$

To each coordinate system in which the Schrödinger equation is separable, there corresponds a set of two integrals of motion. Hence, for the Hamiltonian (3.2), there are four such sets $\{W_1, W_2\}$, $\{X_1, X_2\}$, $\{Y_1, Y_2\}$ and $\{Z_1, Z_2\}$. It turns out, however, that only three of these eight operators are independent, namely the operators W_1, $W_2 = X_1$, and X_2, defined in (A1.9), (3.17), and (3.18), respectively. Since

$$[W_1, X_1] = [X_1, X_2] = 0 \qquad [W_1, X_2] \neq 0 \tag{3.5}$$

they will give rise to a non-Abelian dynamical invariance algebra, thence to 'accidental' degeneracies. Since it is easier to analyse the latter in circular cylindrical coordinates, we shall now proceed to solve the Schrödinger equation in such coordinates. Its solution in the remaining three coordinate systems is given in appendices 1 and 2.

In circular cylindrical coordinates ρ, z, φ ($0 \leqslant \rho < +\infty$, $-\infty < z < +\infty$, $0 \leqslant \varphi < 2\pi$), the Schrödinger equation (3.1) can be written as

$$\frac{1}{2}\left[-\frac{1}{\rho}\partial_\rho \rho \partial_\rho - \frac{1}{\rho^2}\partial^2_{\varphi\varphi} - \partial^2_{zz} + \gamma^2\left(\rho^2 + z^2 + \frac{q\eta^4}{\rho^2}\right)\right]\psi(\rho, z, \varphi) = E\psi(\rho, z, \varphi). \tag{3.6}$$

It separates into the following three differential equations:

$$(\mathrm{d}^2_{\varphi\varphi} + m^2)\Phi(\varphi) = 0 \tag{3.7}$$

$$[\mathrm{d}^2_{zz} - \gamma^2 z^2 + (2\nu + 1)\gamma]\mathscr{Z}(z) = 0 \tag{3.8}$$

$$\left(\frac{1}{\rho}\mathrm{d}_\rho \rho \mathrm{d}_\rho - \frac{M^2}{\rho^2} - \gamma^2\rho^2 + 2E - (2\nu + 1)\gamma\right)\mathscr{R}(\rho) = 0 \tag{3.9}$$

where m^2 and ν are two separation constants, and we have set

$$\psi(\rho, z, \varphi) = \mathscr{R}(\rho)\mathscr{Z}(z)\Phi(\varphi) \tag{3.10}$$

and

$$|M| = (m^2 + q\eta^4\gamma^2)^{1/2}. \tag{3.11}$$

Single-valued, square-integrable solutions of (3.7)-(3.9) are obtained for $m \in \mathbb{Z}$, $\nu \in \mathbb{N}$, and

$$E_N = (N + \tfrac{3}{2})\gamma \tag{3.12}$$

where

$$N = 2n + \nu + |M| \qquad n, \nu \in \mathbb{N}. \tag{3.13}$$

They are given by

$$\Phi_m(\varphi) = (2\pi)^{-1/2}\exp(\mathrm{i}m\varphi) \tag{3.14}$$

$$\mathscr{Z}_\nu(z) = \left(\frac{\sqrt{\gamma}}{\sqrt{\pi}2^\nu \nu!}\right)^{1/2} \exp(-\tfrac{1}{2}\gamma z^2)H_\nu(z\sqrt{\gamma}) \tag{3.15}$$

and

$$\mathscr{R}_{n|M|}(\rho) = \left(\frac{2\gamma n!}{\Gamma(n + |M| + 1)}\right)^{1/2} \exp(-\tfrac{1}{2}\gamma\rho^2)(\rho\sqrt{\gamma})^{|M|}L_n^{|M|}(\gamma\rho^2) \tag{3.16}$$

where H_ν and $L_n^{|M|}$ are Hermite and Laguerre polynomials, respectively (Abramowitz and Stegun 1965).

The two integrals of motion can be written as (Makarov *et al* 1967)

$$X_1 = L_z^2 + q\eta^4\gamma^2 = -\partial^2_{\varphi\varphi} + q\eta^4\gamma^2 \tag{3.17}$$

and

$$X_2 = p_z^2 + \gamma^2 z^2 = -\partial^2_{zz} + \gamma^2 z^2 \tag{3.18}$$

where L_z and p_z are the z components of the angular and linear momenta, respectively. The wavefunctions (3.10), henceforth denoted by

$$\psi_{n\nu m}(\rho, z, \varphi) = \langle \rho z \varphi | n\nu m \rangle \tag{3.19}$$

are the common eigenfunctions of the complete set of commuting operators $\{H, X_1, X_2\}$, corresponding to the eigenvalues E_N, M^2 and $(2\nu+1)\gamma$, respectively.

In the next section we shall proceed to analyse the energy spectrum and show that its 'accidental' degeneracies are due to an su(2) dynamical invariance algebra.

4. Accidental degeneracies and dynamical invariance algebra

The energy spectrum, as given by (3.12) and (3.13), obviously exhibits 'accidental' degeneracies. We shall only be concerned here with the degeneracies $d(N, m)$ for fixed values of N and m, i.e. those degeneracies associated with the two-dimensional Hamiltonian

$$H_{|M|} = H_1 + H_2 \tag{4.1}$$

where

$$H_1 = \tfrac{1}{2}\left(p_\rho^2 - \frac{\mathrm{i}}{\rho} p_\rho + \frac{M^2}{\rho^2} + \gamma^2 \rho^2 \right) \tag{4.2}$$

$$H_2 = \tfrac{1}{2}(p_z^2 + \gamma^2 z^2) \tag{4.3}$$

and $p_\rho = -\mathrm{i}\partial_\rho$, $p_z = -\mathrm{i}\partial_z$ are the momenta canonically conjugate to ρ and z, respectively. As for the Hartmann potential, whenever the angular momentum component m is different from zero, there is an extra degeneracy connected with the two values $|m|$ and $-|m|$.

From (3.12) and (3.13), it follows that

$$d(N, m) = \tfrac{1}{2}(N - |M| - \sigma) + 1 \tag{4.4}$$

where σ is defined by

$$N - |M| = \sigma \bmod 2. \tag{4.5}$$

As shown in figure 1, the levels with $\sigma = 0$ (or $N - |M|$ even) have the same spectrum and degeneracies as a two-dimensional harmonic oscillator of frequency 2γ, and the same is true for the levels with $\sigma = 1$ (or $N - |M|$ odd). Hence, we shall divide the Hilbert space spanned by the states $|n\nu m\rangle$ with a fixed m value into two subspaces S_σ, $\sigma = 0, 1$, whose basis states are defined by

$$|k_1 k_2 m \sigma) = |n\nu m\rangle \qquad k_1, k_2 \in \mathbb{N} \tag{4.6}$$

where

$$k_1 = n \qquad k_2 = \tfrac{1}{2}(\nu - \sigma) \tag{4.7}$$

and, on the left-hand side of (4.6), we use a round bracket notation.

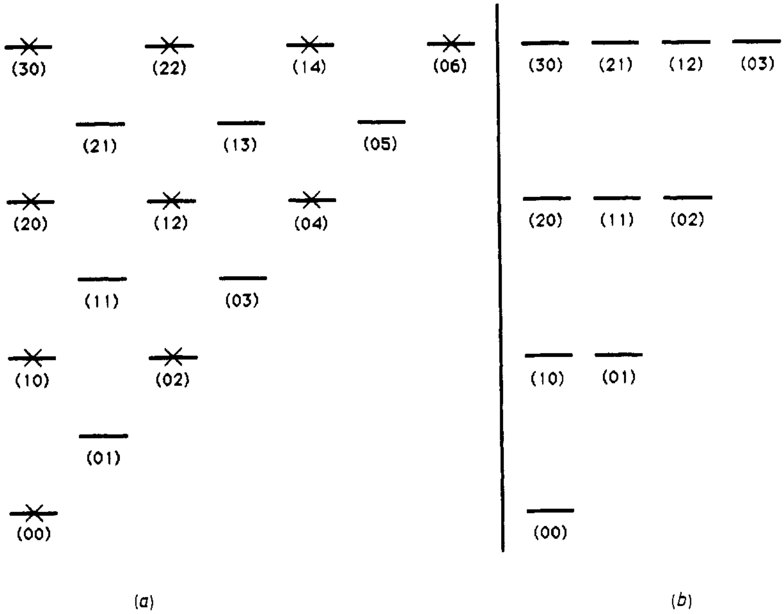

Figure 1. The energy levels of the ring-shaped Hamiltonian (*a*) against those of a two-dimensional harmonic oscillator (*b*) of frequency 2γ. The former are labelled by $(n\nu)$, and the latter by (k_1k_2). The levels of the ring-shaped Hamiltonian with $\sigma=0$ $(\sigma=1)$ are marked by a cross (unmarked), and can be put into one-to-one correspondence with those of the oscillator.

In each subspace S_σ, let us construct the operators $A^\dagger_{\sigma i}$, $A_{\sigma i}$, $i=1,2$, whose action on the basis states (4.6) is that of boson creation and annihilation operators, i.e.

$$\begin{aligned} A^\dagger_{\sigma 1}|k_1k_2m\sigma\rangle &= (k_1+1)^{1/2}|k_1+1\ k_2\ m\ \sigma\rangle \\ A_{\sigma 1}|k_1k_2m\sigma\rangle &= k_1^{1/2}|k_1-1\ k_2\ m\ \sigma\rangle \end{aligned} \tag{4.8}$$

and similar relations for $A^\dagger_{\sigma 2}$ and $A_{\sigma 2}$. Going back to the old notation $|n\nu m\rangle$, these operators have to satisfy the following relations:

$$\begin{aligned} &A^\dagger_{\sigma 1}|n\nu m\rangle = (n+1)^{1/2}|n+1\ \nu\ m\rangle \qquad A_{\sigma 1}|n\nu m\rangle = n^{1/2}|n-1\ \nu\ m\rangle \\ &A^\dagger_{\sigma 2}|n\nu m\rangle = [\tfrac{1}{2}(\nu-\sigma+2)]^{1/2}|n\ \nu+2\ m\rangle \\ &A_{\sigma 2}|n\nu m\rangle = [\tfrac{1}{2}(\nu-\sigma)]^{1/2}|n\ \nu-2\ m\rangle. \end{aligned} \tag{4.9}$$

From these relations, it follows that $A^\dagger_{\sigma 1}$ and $A_{\sigma 1}$ will have the same form in both subspaces S_0 and S_1, whereas $A^\dagger_{\sigma 2}$ and $A_{\sigma 2}$ will explicitly depend on σ.

Let us start with the construction of $A^\dagger_{\sigma 1}$ and $A_{\sigma 1}$. From the recursion and differential relations satisfied by Laguerre polynomials (Abramowitz and Stegun 1965), it follows that the operators $B^\dagger$ and B, defined by

$$\begin{aligned} B^\dagger &= \frac{1}{4\gamma}p_\rho^2 + \frac{\mathrm{i}}{2}\left(\rho - \frac{1}{2\gamma\rho}\right)p_\rho + \frac{M^2}{4\gamma\rho^2} - \tfrac{1}{4}\gamma\rho^2 + \tfrac{1}{2} \\ B &= \frac{1}{4\gamma}p_\rho^2 - \frac{\mathrm{i}}{2}\left(\rho + \frac{1}{2\gamma\rho}\right)p_\rho + \frac{M^2}{4\gamma\rho^2} - \tfrac{1}{4}\gamma\rho^2 - \tfrac{1}{2} \end{aligned} \tag{4.10}$$

satisfy the relations

$$\begin{aligned} B^{\dagger}|n\nu m\rangle &= [(n+1)(n+|M|+1)]^{1/2}|n+1\ \nu\, m\rangle \\ B|n\nu m\rangle &= [n(n+|M|)]^{1/2}|n-1\ \nu\, m\rangle. \end{aligned} \tag{4.11}$$

By noting that

$$\gamma^{-1}H_1|n\nu m\rangle = [B, B^{\dagger}]|n\nu m\rangle = (2n+|M|+1)|n\nu m\rangle \tag{4.12}$$

we obtain the following results for $A^{\dagger}_{\sigma 1}$ and $A_{\sigma 1}$:

$$\begin{aligned} A^{\dagger}_{\sigma 1} &= (2\gamma)^{1/2}B^{\dagger}[H_1+(|M|+1)\gamma]^{-1/2} \\ A_{\sigma 1} &= (2\gamma)^{1/2}B[H_1+(|M|-1)\gamma]^{-1/2}. \end{aligned} \tag{4.13}$$

Let us next construct $A^{\dagger}_{\sigma 2}$ and $A_{\sigma 2}$. In terms of the operators

$$a^{\dagger}_z = (2\gamma)^{-1/2}(\gamma z - ip_z) \qquad a_z = (2\gamma)^{-1/2}(\gamma z + ip_z) \tag{4.14}$$

and H_2, satisfying the relations

$$\begin{aligned} & a^{\dagger}_z|n\nu m\rangle = (\nu+1)^{1/2}|n\ \nu+1\ m\rangle \qquad a_z|n\nu m\rangle = \nu^{1/2}|n\ \nu-1\ m\rangle \\ & \gamma^{-1}H_2|n\nu m\rangle = (a^{\dagger}_z a_z + \tfrac{1}{2})|n\nu m\rangle = (\nu+\tfrac{1}{2})|n\nu m\rangle \end{aligned} \tag{4.15}$$

we get

$$\begin{aligned} A^{\dagger}_{\sigma 2} &= \sqrt{\gamma}(a^{\dagger}_z)^2[2H_2+(2\sigma+1)\gamma]^{-1/2} \\ A_{\sigma 2} &= \sqrt{\gamma}(a_z)^2[2H_2+(2\sigma-3)\gamma]^{-1/2}. \end{aligned} \tag{4.16}$$

From (4.9), it follows that

$$\left(2\gamma \sum_{i=1}^{2} A^{\dagger}_{\sigma i}A_{\sigma i} + \gamma(|M|+\sigma+\tfrac{3}{2})\right)|n\nu m\rangle = E_N|n\nu m\rangle \tag{4.17}$$

whenever $|n\nu m\rangle \in S_\sigma$. The operator on the left-hand side of (4.17) may therefore be identified with the restriction $H_{|M|\sigma}$ of $H_{|M|}$ to S_σ:

$$H_{|M|\sigma} = 2\gamma \sum_{i=1}^{2} A^{\dagger}_{\sigma i}A_{\sigma i} + \gamma(|M|+\sigma+\tfrac{3}{2}). \tag{4.18}$$

Apart from an irrelevant additive constant, $H_{|M|\sigma}$ has been converted into the Hamiltonian of a two-dimensional oscillator of frequency 2γ, hence it has an su(2) dynamical invariance algebra, generated by the operators

$$J_{\sigma+} = A^{\dagger}_{\sigma 1}A_{\sigma 2} \qquad J_{\sigma 0} = \tfrac{1}{2}(A^{\dagger}_{\sigma 1}A_{\sigma 1} - A^{\dagger}_{\sigma 2}A_{\sigma 2}) \qquad J_{\sigma -} = A^{\dagger}_{\sigma 2}A_{\sigma 1}. \tag{4.19}$$

Their action on the basis states of S_σ is given by

$$\begin{aligned} J_{\sigma+}|n\nu m\rangle &= [\tfrac{1}{2}(n+1)(\nu-\sigma)^{1/2}|n+1\ \nu-2\ m\rangle \\ J_{\sigma 0}|n\nu m\rangle &= \tfrac{1}{2}[n-\tfrac{1}{2}(\nu-\sigma)]|n\nu m\rangle \\ J_{\sigma -}|n\nu m\rangle &= [\tfrac{1}{2}n(\nu-\sigma+2)]^{1/2}|n-1\ \nu+2\ m\rangle. \end{aligned} \tag{4.20}$$

All such states with a given N value belong to a single su(2) irreducible representation, characterised by $j = \tfrac{1}{2}[n+\tfrac{1}{2}(\nu-\sigma)] = \tfrac{1}{4}(N-|M|-\sigma)$, and they can be distinguished by the eigenvalue $m_j = \tfrac{1}{2}[n-\tfrac{1}{2}(\nu-\sigma)]$ of J_0.

Let us now go back to the Hilbert space spanned by the whole set of eigenstates of $H_{|M|}$, and introduce projection operators P_σ onto its subspaces S_σ. The operators

$$J_+ = \sum_{\sigma=0}^{1} J_{\sigma+} P_\sigma \qquad J_0 = \sum_{\sigma=0}^{1} J_{\sigma 0} P_\sigma \qquad J_- = \sum_{\sigma=0}^{1} J_{\sigma-} P_\sigma \tag{4.21}$$

leave $H_{|M|}$ invariant, connect all its degenerate eigenstates and satisfy the commutation relations and Hermiticity properties of su(2) generators. Hence, they are the generators of the searched for su(2) dynamical invariance algebra of $H_{|M|}$.

As a final point, let us comment on the classical limit of the transformation defined in (4.13) and (4.16). Such a classical limit can be very easily written down by going back from dimensionless units to normal ones, and letting $\hbar$ go to zero while M goes to infinity. The σ-dependent terms then become negligibly small. By defining

$$Q_i = \tfrac{1}{2}\gamma^{-1/2}(A_i^\dagger + A_i) \qquad P_i = \mathrm{i}\gamma^{1/2}(A_i^\dagger - A_i) \qquad i = 1, 2 \tag{4.22}$$

and taking (4.2), (4.3), (4.10) and (4.14) into account, we obtain

$$\begin{aligned}
Q_1 &= \frac{1}{2\gamma}\left(p_\rho^2 + \frac{M^2}{\rho^2} - \gamma^2\rho^2\right)\left(p_\rho^2 + \frac{M^2}{\rho^2} + \gamma^2\rho^2 + 2|M|\gamma\right)^{-1/2} \\
P_1 &= -2\gamma\rho p_\rho\left(p_\rho^2 + \frac{M^2}{\rho^2} + \gamma^2\rho^2 + 2|M|\gamma\right)^{-1/2} \\
Q_2 &= (2\gamma)^{-1}(-p_z^2 + \gamma^2 z^2)(p_z^2 + \gamma^2 z^2)^{-1/2} \\
P_2 &= 2\gamma z p_z (p_z^2 + \gamma^2 z^2)^{-1/2}.
\end{aligned} \tag{4.23}$$

Equation (4.23) defines a non-bijective canonical transformation between the two phase spaces (ρ, p_ρ, z, p_z) and (Q_1, P_1, Q_2, P_2), such that the classical counterpart of (4.1),

$$H_{|M|} = \tfrac{1}{2}(p_\rho^2 + M^2/\rho^2 + \gamma^2\rho^2) + \tfrac{1}{2}(p_z^2 + \gamma^2 z^2) \tag{4.24}$$

is mapped onto the two-dimensional oscillator Hamiltonian

$$\mathscr{H} = \tfrac{1}{2}\sum_i (P_i^2 + 4\gamma^2 Q_i^2). \tag{4.25}$$

This mapping is two to one since both points (ρ, p_ρ, z, p_z) and $(\rho, p_\rho, -z, -p_z)$ in the original phase space correspond to the same point (Q_1, P_1, Q_2, P_2) in the new one. The ambiguity group (Moshinsky and Seligman 1978, 1979) of the canonical transformation (4.23) contains the unit element and the transformation $(\rho, p_\rho, z, p_z) \to (\rho, p_\rho, -z, -p_z)$. It has two inequivalent one-dimensional irreducible representations, which may be labelled by the index $\sigma = 0, 1$, introduced in (4.6), and known as the ambiguity spin. Bijectiveness may be restored either by introducing two Riemann sheets in the new phase space, or by keeping a single-sheet structure but mapping functions $f(\rho, p_\rho, z, p_z)$ onto two-component functions

$$\begin{bmatrix} F_0(Q_1, P_1, Q_2, P_2) \\ F_1(Q_1, P_1, Q_2, P_2) \end{bmatrix}$$

labelled by the ambiguity spin σ.

In quantum mechanics, the unitary representation of the canonical transformation (4.23) may also be labelled by the ambiguity spin as follows:

$$\langle \rho z | U | Q_1 Q_2 \sigma \rangle = \sum_{k_1, k_2 = 0}^{\infty} \langle \rho z | k_1 k_2 m \sigma) \{k_1 k_2 | Q_1 Q_2 \rangle \tag{4.26}$$

where $|k_1k_2m\sigma)$ is defined in (4.6), and $|k_1k_2\}$ denotes an eigenstate of the two-dimensional oscillator Hamiltonian of frequency 2γ.

5. Conclusion

In the present paper, we have introduced a new ring-shaped potential, that we may call a ring-shaped oscillator, and we have determined its dynamical invariance algebra. The latter is identical with that of the Hartmann potential, as obtained by Kibler and Winternitz (1987). Hence this establishes a close connection between both ring-shaped potentials.

Such a link is not in itself surprising, since many relations are known to exist between the Coulomb and oscillator problems. For instance, they can be related by a canonical transformation (Moshinsky *et al* 1972, Moshinsky and Seligman 1981), and the Coulomb problem can also be reformulated in terms of a constrained four-dimensional oscillator (Kibler and Négadi 1983a, b, 1984a).

However, the addition of a non-central potential of the type $(r\sin\theta)^{-2}$ makes this connection still deeper by smoothing out many discrepancies between both problems. Starting with two distinct dynamical invariance algebras, namely so(4) and su(3), we indeed end up with the same su(2) algebra. The only difference left is that the Hartmann Hamiltonian spectrum can be mapped in a one-to-one fashion onto a harmonic oscillator one, whereas the corresponding mapping is two to one for the ring-shaped oscillator.

Further investigation of the relations between both ring-shaped potentials is in progress. We hope to report on it in a forthcoming publication.

Appendix 1. Schrödinger equation in spherical coordinates

In spherical coordinates r, θ, $\varphi(0\leqslant r<+\infty,\ 0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi<2\pi)$, the Schrödinger equation (3.1) can be written as

$$\frac{1}{2}\left[-\frac{1}{r^2}\partial_r r^2\partial_r-\frac{1}{r^2\sin\theta}\partial_\theta\sin\theta\,\partial_\theta-\frac{1}{r^2\sin^2\theta}\partial^2_{\varphi\varphi}+\gamma^2\left(r^2+\frac{q\eta^4}{r^2\sin^2\theta}\right)\right]\psi(r,\theta,\varphi)=E\psi(r,\theta,\varphi). \tag{A1.1}$$

It separates into (3.7) and the following two differential equations:

$$\left(\frac{1}{\sin\theta}\mathrm{d}_\theta\sin\theta\,\mathrm{d}_\theta-\frac{M^2}{\sin^2\theta}+L(L+1)\right)\Theta(\theta)=0 \tag{A1.2}$$

$$\left(\frac{1}{r^2}\mathrm{d}_r r^2\mathrm{d}_r-\frac{L(L+1)}{r^2}-\gamma^2r^2+2E\right)R(r)=0 \tag{A1.3}$$

where M^2 is defined in (3.11), and we have set

$$\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi). \tag{A1.4}$$

The separation constants are now m^2 and $L(L+1)$.

Single-valued, square-integrable solutions of (3.7), (A1.2), and (A1.3) are obtained for $m \in \mathbb{Z}$,

$$L = \nu + |M| \qquad \nu \in \mathbb{N} \tag{A1.5}$$

and E given by (3.12), where

$$N = 2n + L \qquad n \in \mathbb{N}. \tag{A1.6}$$

They are given by (3.14), and the following relations:

$$\Theta_{L|M|}(\theta) = \left(\frac{(2L+1)(L-|M|)!}{2\Gamma(L+|M|+1)}\right)^{1/2} P_L^{|M|}(\cos\theta) \tag{A1.7}$$

$$R_{nL}(r) = \left(\frac{2\gamma^{3/2} n!}{\Gamma(L+n+\frac{3}{2})}\right)^{1/2} \exp(-\tfrac{1}{2}\gamma r^2)(r\sqrt{\gamma})^L L_n^{(L+1/2)}(\gamma r^2) \tag{A1.8}$$

where $P_L^{|M|}$ is an associated Legendre function of the first kind, and $L_n^{(L+1/2)}$ a Laguerre polynomial (Abramowitz and Stegun 1965).

The integrals of motion are now (Makarov *et al* 1967)

$$W_1 = \boldsymbol{L}^2 + \frac{q\eta^4\gamma^2}{\sin^2\theta} = -\frac{1}{\sin\theta}\partial_\theta \sin\theta\, \partial_\theta - \frac{1}{\sin^2\theta}(\partial^2_{\varphi\varphi} - q\eta^4\gamma^2) \tag{A1.9}$$

and

$$W_2 = X_1 \tag{A1.10}$$

where $\boldsymbol{L}^2$ is the square of the angular momentum. The wavefunctions (A1.4), to be denoted by $\psi_{n\nu m}(r, \theta, \varphi)$, are the common eigenfunctions of the set of commuting operators $\{H, W_1, W_2\}$, corresponding to the eigenvalues E_N, $L(L+1)$ and M^2, respectively. Although, for simplicity, we use here the same notation for the quantum numbers as in § 3, one should bear in mind that n and ν have a different meaning in each case.

Appendix 2. Schrödinger equation in prolate or oblate spheroidal coordinates

In prolate spheroidal coordinates u, v, φ $(1 \leq u < +\infty, -1 \leq v \leq 1, 0 \leq \varphi < 2\pi)$, defined by

$$x = a[(u^2-1)(1-v^2)]^{1/2}\cos\varphi \qquad y = a[(u^2-1)(1-v^2)]^{1/2}\sin\varphi \qquad z = auv \tag{A2.1}$$

the Schrödinger equation (3.1) is

$$\begin{aligned}\frac{1}{2(u^2-v^2)}\bigg[&-\frac{1}{a^2}\left(\partial_u(u^2-1)\partial_u + \partial_v(1-v^2)\partial_v + \frac{u^2-v^2}{(u^2-1)(1-v^2)}\partial^2_{\varphi\varphi}\right)\\ &+ a^2\gamma^2[u^2(u^2-1)+v^2(1-v^2)] + \frac{q\eta^4\gamma^2}{a^2}\left(\frac{1}{u^2-1}+\frac{1}{1-v^2}\right)\bigg]\psi(u,v,\varphi)\\ &= E\psi(u,v,\varphi).\end{aligned} \tag{A2.2}$$

By setting

$$\psi(u, v, \varphi) = U(u)V(v)\Phi(\varphi) \tag{A2.3}$$

it separates into (3.7) for $\Phi(\varphi)$, the following equation for $U(u)$:

$$\left(\mathrm{d}_u(u^2-1)\mathrm{d}_u - \frac{M^2}{u^2-1} - a^4\gamma^2u^2(u^2-1) + 2a^2E(u^2-1) - \zeta\right)U(u) = 0 \tag{A2.4}$$

and a formally identical equation for $V(v)$. The separation constants are now m^2 and ζ, and M^2 is defined in (3.11).

While $\Phi(\varphi)$ is again given by (3.14), where $m \in \mathbb{Z}$, $U(u)$ and $V(v)$ are obtained in the following form (Demeur and Reidemeister 1970):

$$U_{n\nu i|M|}(u) = \exp(-\tfrac{1}{2}\gamma a^2 u^2) u^{\nu} (u^2-1)^{|M|/2} \sum_{s=0}^{n} c_s(n\nu i|M|)(u^2-1)^s \tag{A2.5}$$

$$V_{n\nu i|M|}(v) = \exp(-\tfrac{1}{2}\gamma a^2 v^2) v^{\nu} (1-v^2)^{|M|/2} \sum_{s=0}^{n} c_s(n\nu i|M|)(v^2-1)^s \tag{A2.6}$$

where n, $\nu \in \mathbb{N}$ and i runs over $1, 2, \ldots, n+1$. In (A2.5) and (A2.6), the coefficients $c_s(n\nu i|M|)$ are solutions of the recursion relation

$$A_s c_{s+1} + B_s c_s + C_s c_{s-1} = 0 \qquad s = 0, 1, \ldots, n \tag{A2.7}$$

where $c_{-1} = 0$, and

$$\begin{aligned} A_s &= 4(s+1)(s+|M|+1) \\ B_s &= 4s(s+|M|+\nu+\tfrac{1}{2}-\gamma a^2) + (|M|+1)(|M|+2\nu-2\gamma a^2) - \zeta \\ C_s &= 2\gamma a^2(\gamma^{-1}E - 2s - |M| - \nu + \tfrac{1}{2}). \end{aligned} \tag{A2.8}$$

The energy eigenvalues, given in (3.12) and (3.13), result from the square-integrability condition on $U(u)$, imposing that $C_{n+1} = 0$ for some $n \in \mathbb{N}$, while the separation constant values $\zeta^{(i)}$, $i = 1, \ldots, n+1$, come from the compatibility condition of (A2.7).

Two integrals of motion are now (Makarov *et al* 1967)

$$\begin{aligned} Y_1 &= L^2 - a^2(p_x^2 + p_y^2) - a^4\gamma^2(u^2-1)(1-v^2) + q\eta^4\gamma^2 \frac{u^2+v^2-2}{(u^2-1)(1-v^2)} \\ &= \frac{1}{u^2-v^2}\bigg[(1-v^2)\partial_u(u^2-1)\partial_u - (u^2-1)\partial_v(1-v^2)\partial_v + \left(\frac{1}{u^2-1} - \frac{1}{1-v^2}\right) \\ &\quad \times (u^2-v^2)(\partial^2_{\varphi\varphi} - q\eta^4\gamma^2) - a^4\gamma^2(u^2-v^2)(u^2-1)(1-v^2)\bigg] \\ &= W_1 - 2a^2 H + a^2 X_2 \end{aligned} \tag{A2.9}$$

and

$$Y_2 = L_z^2 + \frac{q\eta^4\gamma^2}{a^2} = X_1 + \left(\frac{1}{a^2} - 1\right) q\eta^4\gamma^2. \tag{A2.10}$$

The wavefunctions (A2.3), to be denoted by $\psi_{n\nu i m}(u, v, \varphi)$, are the common eigenfunctions of the complete set of commuting operators $\{H, Y_1, Y_2\}$, corresponding to the eigenvalues E_N, $\zeta^{(i)}$ and $M^2 + (a^{-2} - 1)q\eta^4\gamma^2$.

In the case of oblate spheroidal coordinates u, v, $\varphi (0 \leqslant u < +\infty$, $-1 \leqslant v \leqslant +1$, $0 \leqslant \varphi < 2\pi)$, defined by

$$\begin{aligned} x &= a[(u^2+1)(1-v^2)]^{1/2} \cos\varphi \qquad y = a[(u^2+1)(1-v^2)]^{1/2} \sin\varphi \\ z &= auv \end{aligned} \tag{A2.11}$$

the Schrödinger equation and the wavefunctions can be obtained from the corresponding results in prolate spheroidal coordinates by substituting iu for u, and $-ia$ for a. Hence we shall not detail them here. Note however that, the range of u being different in both coordinate systems, the wavefunction normalisation will be affected by the substitution. The two integrals of motion are now (Makarov *et al* 1967)

$$Z_1 = W_1 + 2a^2H - a^2X_2 \tag{A2.12}$$

and

$$Z_2 = Y_2 \tag{A2.13}$$

and their eigenvalues are again $\zeta^{(i)}$ and $M^2 + (a^{-2} - 1)q\eta^4\gamma^2$.

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